Collision of two solitary waves for the Zakharov-Kuznetsov equation

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TROND MOHN STIFTELSE

1. Introduction to the ZK equation
2. Particular solutions
   - Solitary waves
   - Multi-solitary waves
3. Description along the time
   - Collision phenomenon
   - Stability
   - Main result

Zakharov-Kuznetsov equation

$$\partial_t u + \partial_x \left(\Delta u + u^2 \right) = 0, \quad u : I_t \times \mathbb{R}^d_{x,y} \to \mathbb{R}, \quad \Delta = \partial_x^2 + \partial_y^2. \quad \text{(ZK)}$$

Here, we consider $d = 2$. $d = 3$: first physical derivation.
$d = 1$: KdV, completely integrable.

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Conserved quantities (at least):

$$\int_{\mathbb{R}^2} u(t,x,y)dxdy, \quad M(u(t)) = \int_{\mathbb{R}^2} \frac{u(t,x,y)^2}{2} dxdy,$$
$$E(u(t)) = \int_{\mathbb{R}^2} \frac{|\nabla u(t,x,y)|^2}{2} - \frac{u(t,x,y)^3}{3} dxdy.$$

Operators that leave invariant the set of solutions u:

- Time translations: $t_0 \in \mathbb{R}$, $u \mapsto u(\cdot - t_0, \cdot, \cdot)$.
- Space translations: $(x_0, y_0) \in \mathbb{R}^2$, $u \mapsto u(\cdot, \cdot - x_0, \cdot - y_0)$
- Scaling: for $\lambda > 0$, $u \mapsto u_\lambda = \lambda u(\lambda\ \ t, \lambda\ \ x, \lambda\ \ y)$.

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- Global : H^1 (Faminskiĭ 1995);
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- Local : H^s, $s > \frac{1}{2}$ (Molinet and Pilod 2015), (Grünrock and Herr 2014); H^s, $s > -\frac{1}{4}$ (Kinoshita 2021)

Solitary waves of velocity $c > 0$ and shift $(x_0, y_0) \in \mathbb{R}^2$ have the form $u(t, x, y) \mapsto Q_c(x - ct - x_0, y - y_0)$, $Q_c \to 0$ when $|(x, y)| \to +\infty$. (First direction with ∂_x).
In the previous form, Q_c satisfies the following equation:

$$-cQ_c + \Delta Q_c + Q_c^2 = 0.$$

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$Q := Q_1$, $Q_c(x, y) = cQ(c^{\frac{1}{2}}x, c^{\frac{1}{2}}y)$. We have the existence of solitary waves associated to every ground-states.

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Theorem (V. 2021)

Let $K \in \mathbb{N}$, $0 < c_1 < \cdots < c_K$, $(x_1, y_1), \cdots (x_K, y_K) \in \mathbb{R}^2$. There exists a unique multi-solitary wave u solution of ZK associated to the previous velocities and shifts. More precisely, there exists T_0 such that $u \in \mathcal{C}((-\infty; T_0); H^1)$ such that

$$\lim_{t\to-\infty} \left\| u(t) - \sum_{k=1}^{K} Q_{c_k}(\cdot - c_k t - x_k, \cdot - y_k) \right\|_{H^1} = 0.$$

KdV equation (Eckhaus and Schuur 1983)

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- **Elastic collision** : if after the collisions, the two solitary waves have the same sizes. Typical for completely integrable equations
- Otherwise, **inelastic collision**. For instance (Mizumachi 2003), (Martel and Merle 2011), (Muñoz 2010); (Martel and Merle 2018) (wave equation)

For ZK, physical equation : what can we expect?

What happens to a perturbed solitary wave?

- 1 solitary wave is orbitally stable (de Bouard 1996)
- 1 solitary wave is asymptotically stable (Côte, Muñoz, Pilod, and Simpson 2016)
- Multi-solitary waves are orbitally stable (Côte, Muñoz, Pilod, and Simpson 2016)

Next part: on the blackboard